

LETTER

Quantum Frequency Arrangements, Quantum Mixed Orthogonal Arrays and Entangled States*

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SUMMARY In this work, we introduce notions of quantum frequency arrangements consisting of quantum frequency squares, cubes, hypercubes and a notion of orthogonality between them. We also propose a notion of quantum mixed orthogonal array (QMOA). By using irredundant mixed orthogonal array proposed by Goyeneche et al. we can obtain k -uniform states of heterogeneous systems from quantum frequency arrangements and QMOAs. Furthermore, some examples are presented to illustrate our method.

key words: quantum frequency arrangements, quantum mixed orthogonal array, irredundant orthogonal array, k -uniform states

1. Introduction

The phenomenon of entanglement is a remarkable feature of quantum physics that has been identified as a key ingredient in many areas of quantum information theory including quantum key distribution [4], superdense coding [1], and teleportation [2]. However, the general problem of how to construct genuinely multipartite entangled states remains unresolved. There has been some progress towards a solution [5]–[7], [10], [20], but the task at hand is generally considered a difficult one.

As is often the case [15], [17], combinatorics can be useful to quantum information theory, and orthogonal arrays (OAs) are fundamental ingredients in the construction of other useful combinatorial objects [9]. Recently, many new methods of constructing OAs of strength k , especially mixed orthogonal arrays (MOAs), have been presented, and many new classes of OAs have been obtained [3], [16], [18], [19]. It is these new developments in OAs that suggest the possibility of constructing infinitely many new genuinely multipartite entangled states. A highly entangled quantum state of heterogeneous multipartite systems consisting of $N > 2$ parties is said to be k -uniform if every reduction to k parties is maximally mixed [6]. These states are closely related to quantum error correction codes over mixed alphabets. Recently, quantum Latin squares, cubes, hypercubes, and quantum orthogonal arrays have been introduced by the authors in [8], [11], [12], [22]. They also demonstrated that

k -uniform states constructed from quantum Latin arrangements have high persistency of entanglement, which makes them ideal candidates for quantum information protocols. However, these combinatorial designs were only used to construct k -uniform states of homogeneous systems [8]. Therefore, more mathematical tools need to be discovered to construct k -uniform states of heterogeneous systems in [6].

In this work, we introduce notions of quantum frequency arrangements consisting of quantum frequency squares, cubes, hypercubes and a notion of orthogonality between them. We also propose a notion of quantum mixed orthogonal array (QMOA). By using irredundant mixed orthogonal array which is proposed in [6] we can obtain k -uniform states of heterogeneous systems from quantum frequency arrangements and QMOAs. Furthermore, some examples are presented to illustrate our method.

2. Preliminaries

Let A^T be the transposition of matrix A and $(d) = (0, 1, \dots, d-1)^T$. Let 0_r and 1_r denote the $r \times 1$ vectors of 0 s and 1 s, respectively. If $A = (a_{ij})_{m \times n}$ and $B = (b_{ij})_{u \times v}$ with elements from a Galois field with binary operations ($+$ and \cdot), the Kronecker product $A \otimes B$ is defined as $A \otimes B = (a_{ij} \cdot B)_{mu \times nv}$, where $a_{ij} \cdot B$ represents the $u \times v$ matrix with entries $a_{ij} \cdot b_{rs}$ ($1 \leq r \leq u, 1 \leq s \leq v$). A matrix A can often be identified with a set of its row vectors if necessary. Let $\mathcal{H}_d^{\otimes m}$ be $\underbrace{\mathcal{H}_d \otimes \dots \otimes \mathcal{H}_d}_m$.

Definition 1: ([9]) An orthogonal array $OA(r, N, d_1 d_2 \dots d_N, k)$ is an $r \times N$ matrix, with the property that, in any $r \times k$ submatrix, all possible combinations of k symbols appear equally often as a row. The orthogonal array is called symmetrical if $d_1 = d_2 = \dots = d_N$. Otherwise, the array is called a MOA.

Definition 2: ([6]) A $MOA(r, N, d_1 d_2 \dots d_N, k)$ is called irredundant, written IrMOA, if every subset of $N - k$ columns contains a different sequence of $N - k$ symbols in every row.

Definition 3: ([20]) Let $S^l = \{(v_1, \dots, v_l) | v_i \in S, i = 1, 2, \dots, l\}$. The Hamming distance $HD(u, v)$ between two vectors $u = (u_1, \dots, u_l), v = (v_1, \dots, v_l) \in S^l$ is defined as the number of positions in which they differ. The minimal distance of a matrix A , written $MD(A)$, is defined to be the minimal Hamming distance between its distinct rows.

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HD(A) is used to represent all the values of the Hamming distances between two distinct rows of A .

Definition 4: ([13]) Suppose we have physical systems A and B , whose state is described by a density operator ρ^{AB} . The reduced density operator for system A is defined by

$$\rho_A \equiv \text{Tr}_B(\rho^{AB}),$$

where Tr_B is a map of operators known as the partial trace over system B . The partial trace is defined by

$$\text{Tr}_B(|a_1\rangle\langle a_2| \otimes |b_1\rangle\langle b_2|) \equiv |a_1\rangle\langle a_2| \text{Tr}(|b_1\rangle\langle b_2|),$$

where $|a_1\rangle$ and $|a_2\rangle$ are any two vectors in the state space of A , and $|b_1\rangle$ and $|b_2\rangle$ are any two vectors in the state space of B . The trace operation appearing on the right hand side is the usual trace operation for system B , so $\text{Tr}(|b_1\rangle\langle b_2|) = \langle b_2|b_1\rangle$.

Definition 5: ([8]) A quantum orthogonal array QOA(r, N, d, k) is an arrangement consisting of r rows composed by N -partite normalized pure quantum states $|\varphi_j\rangle \in \mathcal{H}_d^{\otimes N}$, having d internal levels each, such that

$$d^k \sum_{i,j=0}^{r-1} \text{Tr}_{i_1, \dots, i_{N-k}}(|\varphi_i\rangle\langle\varphi_j|) = r \mathbb{I}_{d^k}$$

for every subset of $N - k$ parties $\{i_1, \dots, i_{N-k}\}$.

Definition 6: The $3n$ slices in three directions of a cubic matrix $A = \{a_{\alpha\beta\gamma} \in F_d^w | \alpha, \beta, \gamma \in F_n\}$ can be expressed as frontal slice: $A_{\alpha, :, :} = \{a_{\alpha\beta\gamma} \in F_d^w | \beta, \gamma \in F_n, \alpha \in F_n\}$, lateral slice: $A_{:, \beta, :} = \{a_{\alpha\beta\gamma} \in F_d^w | \alpha, \gamma \in F_n, \beta \in F_n\}$, and horizontal slice: $A_{:, :, \gamma} = \{a_{\alpha\beta\gamma} \in F_d^w | \alpha, \beta \in F_n, \gamma \in F_n\}$.

Lemma 1: A MOA($r, N, d_1 \dots d_N, k$) is irredundant if and only if its minimal distance is greater than k .

Proof. It follows from the definition of IrMOA.

Lemma 2: ([9]) Taking the runs in an $A = \text{OA}(r, N, d_1 d_2 \dots d_N, k)$ that begin with 0 (or any other particular symbol) and omitting the first column yields an $\text{OA}(r/d_1, N - 1, d_2 \dots d_N, k - 1)$, denoted by A_0 . Similarly, we can obtain $A_1, A_2, \dots, A_{d_1-1}$ are all $\text{OA}(r/d_1, N - 1, d_2 \dots d_N, k - 1)$. Then A can be written as follows.

$$A = \begin{pmatrix} 0_{r/d_1} & A_0 \\ 1_{r/d_1} & A_1 \\ \vdots & \vdots \\ (d_1 - 1)_{r/d_1} & A_{d_1-1} \end{pmatrix}.$$

Lemma 3: Let A as in Lemma 2 be an IrMOA($r, N, d_1 d_2 \dots d_N, k$). Then A_i is an IrMOA($r/d_1, N - 1, d_2 \dots d_N, k - 1$) and $\text{MD}(A_i) \geq k + 1$ for $i = 0, 1, \dots, d_1 - 1$.

Proof. It follows from Lemma 2 that A_i is an $\text{OA}(r/d_1, N - 1, d_2 \dots d_N, k - 1)$. By Lemma 1, we have $\text{MD}(A) \geq k + 1$. Then $\text{MD}(A_i) \geq k + 1 > k$. Therefore, A_i is an IrMOA($r/d_1, N - 1, d_2 \dots d_N, k - 1$) by Lemma 1.

Lemma 4: Suppose A is an IrMOA($n^k, m + k, n^k d^m, k$).

Then,

- (1) If $d \leq n$, then $m \geq k$.
- (2) If $d < n$, then $m > k$.
- (3) If $d = n$, then $m \geq k$.

Proof. We only prove the second case. Assume $m \leq k$, then after removing the first k columns of A , the $n^k \times m$ subarray contains two same rows, which is a contradiction.

Unless stated otherwise, we only consider the case of $n > d$.

3. Main Results

Definition 7: A quantum frequency square of size n is an arrangement

$$\text{QFS}(n, d) = \begin{pmatrix} |\psi_{0,0}\rangle & \dots & |\psi_{0,n-1}\rangle \\ \vdots & & \vdots \\ |\psi_{n-1,0}\rangle & \dots & |\psi_{n-1,n-1}\rangle \end{pmatrix}$$

composed of n^2 single-particle quantum states $|\psi_{i,j}\rangle \in \mathcal{H}_d$, $i, j \in \{0, \dots, n - 1\}$, such that each row and each column determine n/d orthonormal bases for a qudit system.

Definition 8: A set of n^2 pure quantum states $|\psi_{i,j}\rangle \in \mathcal{H}_d^{\otimes m}$, $m > 2$ arranged as

$$\begin{pmatrix} |\psi_{0,0}\rangle & \dots & |\psi_{0,n-1}\rangle \\ \vdots & & \vdots \\ |\psi_{n-1,0}\rangle & \dots & |\psi_{n-1,n-1}\rangle \end{pmatrix},$$

forms a set of m mutually orthogonal quantum frequency squares (m MOQFS(n, d)) if the following properties hold:

- (1) The set of n^2 states $\{|\psi_{i,j}\rangle | i, j = 0, \dots, n - 1\}$ are orthogonal.
- (2) The sum of every row in the array, i.e., $\sum_{j=0}^{n-1} |\psi_{i,j}\rangle$, is a 1-uniform state.
- (3) The sum of every column in the array, i.e., $\sum_{i=0}^{n-1} |\psi_{i,j}\rangle$, is a 1-uniform state.

Definition 9: A quantum mixed orthogonal array QMOA($r, N, d_1 d_2 \dots d_N, k$) is an arrangement consisting of r rows composed by N -partite normalized pure quantum states $|\varphi_j\rangle \in \mathcal{H}_{d_1} \otimes \dots \otimes \mathcal{H}_{d_N}$, such that

$$d_{l_{N-k+1}} \dots d_{l_N} \sum_{i,j=0}^{r-1} \text{Tr}_{l_1, \dots, l_{N-k}}(|\varphi_i\rangle\langle\varphi_j|) = r \mathbb{I}_{d_{l_{N-k+1}} \dots d_{l_N}},$$

for every subset of $N - k$ parties $\{l_1, \dots, l_{N-k}\}$.

Theorem 1: (1) From an IrMOA($n^2, m + 2, n^2 d^m, 2$), we can construct a QMOA($n^2, m + 2, n^2 d^m, 2$) and a set of m MOQFS(n, d).

(2) From the set of MOQFS in (1), we can define a QMOA($n^2, m + 2, n^2 d^m, 2$).

(3) From the QMOA in (1), we can generate the set of m MOQFS(n, d).

Proof. From IrMOA($n^2, m + 2, n^2 d^m, 2$) by a sequence of

permutations of the columns, the rows, and the levels of each factor, we can obtain $A = (a, b, C)$, where $a = (n) \otimes 1_n$, $b = 1_n \otimes (n)$, and $C = (c_{ij})_{n^2 \times m}$ for $i = 0, 1, \dots, n^2 - 1$, $j = 3, 4, \dots, m + 2$.

Construction. From A , we can obtain $B = \text{QMOA}(n^2, m + 2, n^2 d^m, 2)$ and a set of m MOQFS(n, d) M as follows.

$$B = \begin{pmatrix} |0\rangle & |0\rangle & |c_{0,3}c_{0,4} \dots c_{0,m+2}\rangle \\ \vdots & \vdots & \vdots \\ |0\rangle & |n-1\rangle & |c_{n-1,3}c_{n-1,4} \dots c_{n-1,m+2}\rangle \\ \vdots & \vdots & \vdots \\ |n-1\rangle & |0\rangle & |c_{(n-1)n,3}c_{(n-1)n,4} \dots c_{(n-1)n,m+2}\rangle \\ \vdots & \vdots & \vdots \\ |n-1\rangle & |n-1\rangle & |c_{n^2-1,3}c_{n^2-1,4} \dots c_{n^2-1,m+2}\rangle \end{pmatrix},$$

$$M = \begin{pmatrix} |c_{0,3} \dots c_{0,m+2}\rangle & \dots & |c_{n-1,3} \dots c_{n-1,m+2}\rangle \\ \vdots & & \vdots \\ |c_{(n-1)n,3} \dots c_{(n-1)n,m+2}\rangle & \dots & |c_{n^2-1,3} \dots c_{n^2-1,m+2}\rangle \end{pmatrix}.$$

Verification. (1) Firstly, we prove B is a $\text{QMOA}(n^2, m + 2, n^2 d^m, 2)$.

Let $|\varphi_i\rangle = |a_i b_i c_{i,3} \dots c_{i,m+2}\rangle$ and $T_{i,j} = \text{Tr}_{l_1, \dots, l_m}(|\varphi_i\rangle\langle\varphi_j|)$, where $i, j = 0, 1, \dots, n^2 - 1$ and $\{l_1, l_2, \dots, l_m, l_{m+1}, l_{m+2}\} = \{1, 2, \dots, m + 2\}$. Since A is an $\text{IrMOA}(n^2, m + 2, n^2 d^m, 2)$, $T_{i,j} = 0$ for $i \neq j$. Therefore, we will consider the following three cases.

Case 1. If $1, 2 \notin \{l_1, l_2, \dots, l_m\}$, then

$$T_{i,i} = \text{Tr}_{3,4, \dots, m+2}(|\varphi_i\rangle\langle\varphi_i|) = \langle c_{i,3}c_{i,4} \dots c_{i,m+2} | c_{i,3}c_{i,4} \dots c_{i,m+2} \rangle |a_i b_i\rangle\langle a_i b_i| = |a_i b_i\rangle\langle a_i b_i|.$$

In the columns (a, b) , each of all possible pairs as a row occurs with frequency $n^2/n^2 = 1$, then $\sum_{i=0}^{n^2-1} T_{i,i} = I_{n^2}$. We have

$$d_{l_{m+1}} d_{l_{m+2}} \sum_{i,j=0}^{n^2-1} T_{i,j} = d_a d_b \sum_{i=0}^{n^2-1} T_{i,i} = n^2 I_{n^2} = n^2 I_{d_{l_{m+1}} d_{l_{m+2}}}.$$

Case 2. If $\{1, 2\} \subseteq \{l_1, l_2, \dots, l_m\}$, we can without loss of generality assume that $\{l_1, l_2\} = \{1, 2\}$. Then

$$T_{i,i} = \text{Tr}_{1,2, \dots, l_m}(|\varphi_i\rangle\langle\varphi_i|) = \langle a_i b_i c_{i,l_3} \dots c_{i,l_m} | a_i b_i c_{i,l_3} \dots c_{i,l_m} \rangle |c_{i,l_{m+1}} c_{i,l_{m+2}}\rangle\langle c_{i,l_{m+1}} c_{i,l_{m+2}}| = |c_{i,l_{m+1}} c_{i,l_{m+2}}\rangle\langle c_{i,l_{m+1}} c_{i,l_{m+2}}|.$$

Since each of all possible pairs as a row occurs n^2/d^2 times in $(c_{l_{m+1}}, c_{l_{m+2}})$, $\sum_{i=0}^{n^2-1} T_{i,i} = n^2/d^2 I_{d^2}$. We have

$$d_{l_{m+1}} d_{l_{m+2}} \sum_{i,j=0}^{n^2-1} T_{i,j} = d^2 \sum_{i=0}^{n^2-1} T_{i,i} = d^2 (n^2/d^2) I_{d^2} = n^2 I_{d^2} = n^2 I_{d_{l_{m+1}} d_{l_{m+2}}}.$$

Case 3. If $1 \in \{l_1, l_2, \dots, l_m\}$ and $2 \notin \{l_1, l_2, \dots, l_m\}$, or $1 \notin \{l_1, l_2, \dots, l_m\}$ and $2 \in \{l_1, l_2, \dots, l_m\}$, here we only prove the first case. Without loss of generality we may assume that $l_1 = 1$ and $l_{m+1} = 2$. Then

$$T_{i,i} = \text{Tr}_{l_1, \dots, l_m}(|\varphi_i\rangle\langle\varphi_i|) = \langle a_i c_{i,l_2} c_{i,l_3} \dots c_{i,l_m} | a_i c_{i,l_2} c_{i,l_3} \dots c_{i,l_m} \rangle |b_i c_{i,l_{m+2}}\rangle\langle b_i c_{i,l_{m+2}}| = |b_i c_{i,l_{m+2}}\rangle\langle b_i c_{i,l_{m+2}}|.$$

Each of all possible pairs as a row in $(b, c_{l_{m+2}})$ occurs $n^2/(nd) = n/d$ times, then $\sum_{i=0}^{n^2-1} T_{i,i} = (n/d) I_{nd}$. Thus

$$d_{l_{m+1}} d_{l_{m+2}} \sum_{i,j=0}^{n^2-1} T_{i,j} = nd \sum_{i=0}^{n^2-1} T_{i,i} = nd \frac{n}{d} I_{nd} = n^2 I_{nd} = n^2 I_{d_{l_{m+1}} d_{l_{m+2}}}.$$

It follows from Definition 9 that B is a $\text{QMOA}(n^2, m + 2, n^2 d^m, 2)$.

Secondly, we prove M is a set of m MOQFS(n, d). Here we need to consider the three properties as follows.

A is an $\text{IrMOA}(n^2, m + 2, n^2 d^m, 2)$, then any two rows of the submatrix $(c_3, c_4, \dots, c_{m+2})$ are different. Therefore, the n^2 quantum states of M correspond to the n^2 rows of A , that is, the n^2 quantum states of M are pairwise orthogonal.

Taking the runs in A that begin with 0 yields a submatrix denoted by A_1 .

$$A_1 = \begin{pmatrix} 0 & 0 & c_{0,3} & c_{0,4} & \dots & c_{0,m+2} \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & n-1 & c_{n-1,3} & c_{n-1,4} & \dots & c_{n-1,m+2} \end{pmatrix} = (a_1, b_1, C_1),$$

where $a_1 = 0_n$ and $b_1 = (n)$. By Lemma 3, (b_1, C_1) is an $\text{IrMOA}(n, m + 1, nd^m, 1)$ with $\text{MD} \geq 3$ and then C_1 is an $\text{IrOA}(n, m, d, 1)$. The sum of the first row in M , namely $|c_{0,3}c_{0,4} \dots c_{0,m+2}\rangle + \dots + |c_{n-1,3}c_{n-1,4} \dots c_{n-1,m+2}\rangle$, is a 1-uniform state. Similarly, the sum of every row in M is a 1-uniform state.

By the same argument, the sum of every column in M is a 1-uniform state.

Therefore, M is a set of m MOQFS(n, d).

(2) Obviously, as the M in (1) is a set of m MOQFS(n, d) constructed from A , we can obtain the B in (1).

(3) We can easily use the B in (1) to generate the M since B is a $\text{QMOA}(n^2, m + 2, n^2 d^m, 2)$ constructed from A .

Example 1: We can construct a $\text{QMOA}(16, m + 2, 4^2 2^m, 2)$ and a set of m MOQFS(4, 2) for $4 \leq m \leq 9$.

Proof. From the $\text{OA}(16, 5, 4, 2)$ in [14] and $\text{OA}(4, 3, 2, 2) =$

$$\begin{pmatrix} 000 \\ 011 \\ 101 \\ 110 \end{pmatrix},$$

then a saturated $\text{MOA}(16, 11, 4^2 2^9, 2)$ can be obtained by using the expansive replacement method as follows. A $\text{MOA}(r, N, d_1 d_2 \dots d_N, 2)$ is called saturated if $\sum_{i=1}^N (d_i - 1) = r - 1$ [14]. $L = \text{MOA}(16, 11, 4^2 2^9, 2) =$

$$(a, b, c_1, c_2, \dots, c_9) = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 3 & 3 & 3 & 3 \\ 0 & 1 & 2 & 3 & 0 & 1 & 2 & 3 & 0 & 1 & 2 & 3 & 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}^T$$

(1) We have $\text{HD}(L) = 6, 7$ by Ref. [21]. Then we can obtain an IrMOA(16, $m + 2, 4^2 2^m, 2$) for $6 \leq m \leq 9$ by Lemma 1. Thus, by Theorem 1 we can construct a QMOA(16, $m + 2, 4^2 2^m, 2$) and a set of m MOQFS(4, 2) for $6 \leq m \leq 9$.

(2) An IrMOA(16, $6, 4^2 2^4, 2$) can be obtained from a MOA(16, 4, 4, 2) and an OA(4, 2, 2, 2) by using the expansive replacement method. Then we can generate a QMOA(16, $6, 4^2 2^4, 2$) and a set of 4 MOQFS(4, 2) by Theorem 1. Additionally, we can add a 2 level column to the IrMOA(16, $6, 4^2 2^4, 2$) to obtain a MOA(16, $7, 4^2 2^5, 2$). It is evident that the MOA(16, $7, 4^2 2^5, 2$) is redundant. Therefore, by Theorem 1, we can construct a QMOA(16, $7, 4^2 2^5, 2$) and a set of 5 MOQFS(4, 2).

Definition 10: A quantum frequency cube QFC(n, d) of size n is a cubic arrangement composed of n^3 single-particle quantum pure states $|\psi_{i,j,k}\rangle \in \mathcal{H}_d$, $i, j, k \in \{0, \dots, n-1\}$, such that every edge (row, column, file) determines n/d orthogonal bases.

Definition 11: A set of n^3 m -qudit pure states $|\psi_{i,j,k}\rangle$, $i, j, k \in \{0, 1, \dots, n-1\}$, belonging to a composed Hilbert space $\mathcal{H}_d^{\otimes m}$, $m > 3$ forms m triplewise orthogonal quantum frequency cubes (m MOQFC($n, d, 3$)) if the following properties hold:

(1) The set of n^3 states are orthogonal.

(2) The sum of every edge in this array, i.e., $\sum_{k=0}^{n-1} |\psi_{i,j,k}\rangle$, $\sum_{j=0}^{n-1} |\psi_{i,j,k}\rangle$, $\sum_{i=0}^{n-1} |\psi_{i,j,k}\rangle$, is a 1-uniform state, respectively.

(3) The $(m+2)$ -qudit quantum states $\sum_{j,k=0}^{n-1} |j\rangle|k\rangle|\psi_{i,j,k}\rangle$ for $i = 0, 1, \dots, n-1$, $\sum_{i,k=0}^{n-1} |i\rangle|k\rangle|\psi_{i,j,k}\rangle$ for $j = 0, 1, \dots, n-1$, and $\sum_{i,j=0}^{n-1} |i\rangle|j\rangle|\psi_{i,j,k}\rangle$ for $k = 0, 1, \dots, n-1$ are 2-uniform states, respectively.

Theorem 2: (1) From an IrMOA($n^3, m+3, n^3 d^m, 3$), we can construct a QMOA($n^3, m+3, n^3 d^m, 3$) and a set of m MOQFC($n, d, 3$).

(2) We can use the QMOA($n^3, m+3, n^3 d^m, 3$) in (1) to generate the set of m MOQFC($n, d, 3$) in (1).

(3) We can define the QMOA($n^3, m+3, n^3 d^m, 3$) in (1) from the set of m MOQFC($n, d, 3$) in (1).

Proof. From an IrMOA($n^3, m+3, n^3 d^m, 3$) by a sequence of permutations of the columns, the rows, and the levels of each factor, we can obtain a matrix $A = (a, b, c, C)$, where $a = (n) \otimes 1_{n^2}$, $b = 1_n \otimes (n) \otimes 1_n$, $c = 1_{n^2} \otimes (n)$ and $C = (c_{ij})_{n^3 \times m}$ for $i = 0, 1, \dots, n^3 - 1$, $j = 4, \dots, m+3$.

Construction. From A , we can obtain an $n^3 \times (m+3)$ matrix B and a cubic matrix M of size m . B and the u^{th} frontal slice of the M are as follows:

$$B = \begin{pmatrix} |0\rangle & |0\rangle & |0\rangle & |c_{0,4} \dots c_{0,m+3}\rangle \\ \vdots & \vdots & \vdots & \vdots \\ |0\rangle & |0\rangle & |n-1\rangle & |c_{n-1,4} \dots c_{n-1,m+3}\rangle \\ \vdots & \vdots & \vdots & \vdots \\ |n-1\rangle & |n-1\rangle & |0\rangle & |c_{(n-1)n,4} \dots c_{(n-1)n,m+3}\rangle \\ \vdots & \vdots & \vdots & \vdots \\ |n-1\rangle & |n-1\rangle & |n-1\rangle & |c_{n^3-1,4} \dots c_{n^3-1,m+3}\rangle \end{pmatrix},$$

$$M_{u,\dots} = \begin{pmatrix} |c_{un^2,4} \dots c_{un^2,m+3}\rangle & \dots & |c_{un^2+n-1,4} \dots c_{un^2+n-1,m+3}\rangle \\ \vdots & & \vdots \\ |c_{(u+1)n^2,4} \dots c_{(u+1)n^2,m+3}\rangle & \dots & |c_{(u+1)n^2+n-1,4} \dots c_{(u+1)n^2+n-1,m+3}\rangle \end{pmatrix}.$$

Then B and M are the QMOA($n^3, m+3, n^3 d^m, 3$) and the set of m MOQFC($n, d, 3$) needed, respectively.

Verification. (1) The proof that B is a QMOA($n^3, m+3, n^3 d^m, 3$) follows from the first part of the proof of the Theorem 1.

Then we prove M is a set of m MOQFC($n, d, 3$). From Theorem 1 and Definition 11, the first two properties hold. Here we consider the third property as follows.

Let A^0 be the submatrix consisting of the rows in A which begin with 0. Then A^0 can be written as

$$A^0 = (a^0, b^0, c^0, C^0) = \begin{pmatrix} 0 & 0 & 0 & c_{0,4} & c_{0,5} & \dots & c_{0,m+3} \\ 0 & 0 & 1 & c_{1,4} & c_{1,5} & \dots & c_{1,m+3} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & n-1 & n-1 & c_{n^2-1,4} & c_{n^2-1,5} & \dots & c_{n^2-1,m+3} \end{pmatrix},$$

where $a^0 = 0_{n^2}$, $b^0 = (n) \otimes 1_n$, $c^0 = 1_n \otimes (n)$, and $C^0 = (c_{ij})$ corresponds to the first frontal slice in M . By Lemma 3, (b^0, c^0, C^0) is an IrMOA($n^2, m+2, n^2 d^m, 2$) with $MD \geq 4$, then $\sum_{j,k=0}^{n-1} |j\rangle|k\rangle|\psi_{i,j,k}\rangle$ is a 2-uniform state for $i = 0$, so is the sum for each $i = 1, \dots, n-1$. Similarly, taking the submatrices consisting of the rows in A with the elements j in the second column or with the elements k in the third column, respectively, we can prove that $\sum_{i,k=0}^{n-1} |i\rangle|k\rangle|\psi_{i,j,k}\rangle$ and $\sum_{i,j=0}^{n-1} |i\rangle|j\rangle|\psi_{i,j,k}\rangle$ are 2-uniform states for every $0 \leq j, k \leq n-1$.

Therefore, M is the set desired.

The proofs of (2) and (3) are similar to the proofs of (2) and (3) in Theorem 1, respectively.

Definition 12: A quantum frequency hypercube with d levels and size n in dimension k , denoted QFH(n, d, k), is an arrangement composed of n^k single-particle quantum states $|\psi_{i_1, \dots, i_k}\rangle \in \mathcal{H}_d$, $i_1, \dots, i_k \in \{0, \dots, n-1\}$, such that all states belonging to an edge of hypercube determine n/d orthogonal bases.

Definition 13: Let $m > k$. A set of m mutually orthogonal quantum frequency hypercubes with d levels and size n in dimension k , namely m MOQFH(n, d, k), is a k -dimensional arrangement composed of n^k m -qudit states $|\psi_{i_1, \dots, i_k}\rangle \in \mathcal{H}_d^{\otimes m}$, $i_1, \dots, i_k \in \{0, \dots, n-1\}$, such that the following properties hold:

(1) The set of n^k states $\{|\psi_{i_1, \dots, i_k}\rangle \in \mathcal{H}_d\}$ are orthogonal.

(2) The sum of n states belonging to the same edge of the hypercube, i.e. $\sum_{i_s=0}^{n-1} |\psi_{i_1, \dots, i_s, \dots, i_k}\rangle$, for every $1 \leq s \leq k$, forms a 1-uniform state.

(3) For any $2 \leq v \leq k-1$ and every subset $\{i_{s_1}, i_{s_2}, \dots, i_{s_v}\} \subseteq \{i_1, \dots, i_k\}$, the sum of the n^v quantum states, denoted by $\sum_{i_{s_1}, i_{s_2}, \dots, i_{s_v}=0}^{n-1} |i_{s_1} i_{s_2} \dots i_{s_v} \psi_{i_1, \dots, i_{s_1}, \dots, i_{s_2}, \dots, i_{s_v}, \dots, i_k}\rangle$, is a v -uniform state.

Theorem 3: From an IrMOA($n^k, m+k, n^k d^m, k$), we can construct a set of m MOQFH(n, d, k) and a QMOA($n^k, m+k, n^k d^m, k$), one of which can be obtained from the other.

Proof: This follows from the arguments analogous to the proof of Theorem 2.

Theorem 4: The sum of rows of a QMOA($r, N, d_1 d_2 \cdots d_N, k$) produces a k -uniform state of a quantum system composed of N parties with d_1, d_2, \dots, d_N levels respectively.

Proof: A positive operator valued measure (POVM) is a set of positive semidefinite operators such that they sum up to identity, determining a generalized quantum measurement [13]. Every reduction to k columns of a QMOA($r, N, d_1 d_2 \cdots d_N, k$) defines a POVM, and thus the sum of its elements produces the identity operator.

Example 2: We can construct a set of 6 MOQFC(4, 2, 3) and a 3-uniform state of a system $4^3 \times 2^6$.

Proof. The frontal slices of the set of 6 MOQFC(4, 2, 3) are as follows.

$$\begin{aligned}
 K_{0,\dots} &= \begin{pmatrix} |000000\rangle & |111101\rangle & |010110\rangle & |101011\rangle \\ |110111\rangle & |001010\rangle & |100001\rangle & |011100\rangle \\ |011001\rangle & |100100\rangle & |001111\rangle & |110010\rangle \\ |101110\rangle & |010011\rangle & |111000\rangle & |000101\rangle \end{pmatrix}, \\
 K_{1,\dots} &= \begin{pmatrix} |011111\rangle & |100010\rangle & |001001\rangle & |110100\rangle \\ |101000\rangle & |010101\rangle & |111110\rangle & |000011\rangle \\ |000110\rangle & |111011\rangle & |010000\rangle & |101101\rangle \\ |110001\rangle & |001100\rangle & |100111\rangle & |011010\rangle \end{pmatrix}, \\
 K_{2,\dots} &= \begin{pmatrix} |100101\rangle & |011000\rangle & |110011\rangle & |001110\rangle \\ |010010\rangle & |101111\rangle & |000100\rangle & |111001\rangle \\ |111100\rangle & |000001\rangle & |101010\rangle & |010111\rangle \\ |001011\rangle & |110110\rangle & |011101\rangle & |100000\rangle \end{pmatrix}, \\
 K_{3,\dots} &= \begin{pmatrix} |111010\rangle & |000111\rangle & |101100\rangle & |010001\rangle \\ |001101\rangle & |110000\rangle & |011011\rangle & |100110\rangle \\ |100011\rangle & |011110\rangle & |110101\rangle & |001000\rangle \\ |010100\rangle & |101001\rangle & |000010\rangle & |111111\rangle \end{pmatrix}.
 \end{aligned}$$

Then by Theorems 2 and 4, we can construct a 3-uniform state of a system $4^3 \times 2^6$ as follows.

$$\begin{aligned}
 |\phi_{4^3 \times 2^6}\rangle &= |000000000\rangle + |001111101\rangle + |002010110\rangle + |003101011\rangle \\ &+ |010110111\rangle + |011001010\rangle + |012100001\rangle + |013011100\rangle \\ &+ |020011001\rangle + |021100100\rangle + |022001111\rangle + |023110010\rangle \\ &+ |030101110\rangle + |031010011\rangle + |032111000\rangle + |033000101\rangle \\ &+ |100011111\rangle + |101100010\rangle + |102001001\rangle + |103110100\rangle \\ &+ |110101000\rangle + |111010101\rangle + |112111110\rangle + |113000011\rangle \\ &+ |120000110\rangle + |121111011\rangle + |122010000\rangle + |123101101\rangle \\ &+ |130110001\rangle + |131001100\rangle + |132100111\rangle + |133011010\rangle \\ &+ |200100101\rangle + |201011000\rangle + |202110011\rangle + |203001110\rangle \\ &+ |210010010\rangle + |211101111\rangle + |212000100\rangle + |213111001\rangle \\ &+ |220111100\rangle + |221000001\rangle + |222101010\rangle + |223010111\rangle \\ &+ |230001011\rangle + |231110110\rangle + |232011101\rangle + |233100000\rangle \\ &+ |300111010\rangle + |301000111\rangle + |302101100\rangle + |303010001\rangle \\ &+ |310001101\rangle + |311110000\rangle + |312011011\rangle + |313100110\rangle \\ &+ |320100011\rangle + |321011110\rangle + |322110101\rangle + |323001000\rangle \\ &+ |330010100\rangle + |331101001\rangle + |332000010\rangle + |333111111\rangle.
 \end{aligned}$$

4. Conclusion

In this letter, we define QMOAs, MOQFS, MOQFC and MOQFH. After setting up the quantum combinatorial tools we present our method for constructing k -uniform states. The further work is to find more QMOAs, MOQFS, MOQFC and MOQFH to construct k -uniform states of heterogeneous systems.

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