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New Construction of Even-Length Binary Z-Complementary Pairs with Low PAPR*

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SUMMARY This paper is focused on constructing even-length binary Z-complementary pairs (EB-ZCPs) with new length. Inspired by a recent work of Adhikary et al., we give a construction of EB-ZCPs with length $8N+4$ (where $N = 2^{\alpha}10^{\beta}26^{\gamma}$ and α, β, γ are nonnegative integers) and zero correlation zone (ZCZ) width $5N+2$. The maximum aperiodic autocorrelation sums (AACS) magnitude of the proposed sequences outside the ZCZ region is 8. It turns out that the generated sequences have low PAPR.

key words: *aperiodic correlation, Golay complementary pair, Z-complementary pairs, PAPR*

1. Introduction

A pair of sequences is called a Golay complementary pair (GCP), if their aperiodic autocorrelation sums (AACSs) are zero everywhere, except at the zero shift [1], [2]. GCPs first introduced by Golay in 1961 in the context of an optical problem in multislit spectrometry, have been used extensively in communication engineering. For instance, a well-known application of GCPs is peak-to-average power ratio (PAPR) control in orthogonal frequency-division multiplexing (OFDM) system [3]–[5]. Besides, GCPs were used for intersymbol interference (ISI) channel estimation [6], [7], and radar waveform design [8]–[11]. Recently, Liu et al. proposed a training waveforms with autocorrelation sidelobes close to zero for continuous phase modulation based on GCPs [12].

The main drawback of the GCPs is their limited availability for various lengths. It is widely believed that binary GCPs can only be found for sequence lengths with form $2^{\alpha}10^{\beta}26^{\gamma}$, where α, β, γ are non-negative integers. This has been verified for binary GCPs of length up to 100 [13]. This motivates the notion of binary Z-complementary pair (ZCP) which was proposed by Fan et al. [14]. It turns out in [14] and [15] that binary ZCPs exist for much more sequence length. It was further conjectured by Fan, Yuan and Tu [14] that "*For EB-ZCPs, given that the lengths* $N \neq 2^{a}10^{b}26^{c}$, *the ZCZ is upper bounded by* $N-2$." Since then, the study of ZCPs has received a lot attention.

In [16], Liu et al. confirmed the above conjecture and gave a systematic construction of the EB-ZCPs based on the head-end (or the tail-end) quarter-sequence-truncation of certain binary Golay-Davis-Jedwab complementary pairs [4] of length 2^{α}. Each EB-ZCP in [16] has length $2^{\alpha+1}+2^{\alpha}$ and zero correlation zone (ZCZ) width of $2^{\alpha+1}$. In [17], Chen also gave a systematic construction of the EB-ZCPs through the generalized Boolean functions, each having length of $2^{m-1}+2^{v}$, and ZCZ width of $2^{m-2}+2^{v}$, where $v \leq m-2, v \in \mathbb{Z}^{+} \cup \{0\}$. In [18], Pai et al. proposed a novel construction of ZCPs based on Boolean functions with flexible length $2^{m-1}+\sum_{\alpha=k+1}^{m-1} a_{\alpha}2^{\alpha-1}+2^{v}$, and ZCZ width of $2^{k-1}+2^{v}$, where $v < k < m$. Recently, Xie et al. proposed a systematic construction of EB-ZCPs [19] with length $28N$ and ZCZ width of $24N$, where $N = 2^{\alpha}10^{\beta}26^{\gamma}$ and α, β, γ are nonnegative integers. Insertion element method is a useful method for constructing ZCPs proposed by Adhikary et al. in [20]. By the method, Adhikary et al. inserted tow elements in a GCP, and then constructed a ZCP with length $4N+2$ and ZCZ width $3N+1$. Very recently, Adhikary et al. combined insertion element method and Boolean functions to construct ZCPs [21]. Inspired this work of Adhikary, we further consider the construction of ZCPs by inserting more elements in GCPs.

In this paper, we present a systematic construction of EB-ZCPs, each having length $8N+4$, and ZCZ width of $5N+2$, i.e., asymptotic ZCZ ratio of $5/8$. Interestingly, each of the constructed EB-ZCPs has identical AACS magnitude of 8 at each time-shift outside the ZCZ (except at the last quarter time-shift, at which the AACS magnitude is zero). The key of the proposed construction is to use some newly intrinsic properties of binary GCPs found in [22]. Table 1 lists the parameters of EB-ZCPs mentioned above.

The rest of the paper is organized as follows. In Sect. 2, we introduce EB-ZCPs and complementary mates. In Sect. 3, we propose asystematic construction of EB-ZCPs of length $8N+4$, each having ZCZ width $5N+2$. We show that outside the ZCZ, the AACS has identical value of 8 except at the last quarter time-shift position. Finally, we conclude this work in Sect. 4.

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Table 1 The parameters of EB-ZCPs.

Length	ZCZ width	Parameter	Rerference
$2^{\alpha+1}+2^{\alpha}$	$2^{\alpha+1}$	$\alpha \in \mathbb{Z}^{+}$	Liu et al. [16]
$2^{m-1}+2^{v}$	$2^{m-2}+2^{v}$	$m \geq 2, v \leq m-2, m, v \in \mathbb{Z}^{+}$	Chen [17]
$2^{m-1}+\sum_{\alpha=k+1}^{m-1} a_{\alpha}2^{\alpha-1}+2^{v}$	$2^{k-1}+2^{v}$	$v < k < k$	Pai et al. [18]
$28N$	$24N$	$N=2^{\alpha}10^{\beta}26^{\gamma}$	Xie et al. [19]
$24N$	$20N$	$N=2^{\alpha}10^{\beta}26^{\gamma}$	Xie et al. [19]
$2^{m+3}+2^{m+2}+2^{m+1}$	2^{m+3}	$m \in \mathbb{Z}^{+}$	Xie et al. [19]
$4N+2$	$3N+1$	$N=2^{\alpha}10^{\beta}26^{\gamma}$	Adhikary et al. [20]
$8N+4$	$5N+2$	$N=2^{\alpha}10^{\beta}26^{\gamma}$	This paper

2. Preliminaries

Throughout this paper, a binary sequence is a vector over signal set $\mathbb{U}=\{+1,-1\}$. Let "$[\mathbf{a},\mathbf{b}]$" denote the horizontal concatenation of sequences $\mathbf{a}$ and $\mathbf{b}$. Also, the reverse of $\mathbf{a}$ is denoted by $\overleftarrow{\mathbf{a}}$, 1 and -1 are denoted by + and −, respectively.

Definition 1. *For two length-N binary sequences $\mathbf{a}$ and $\mathbf{b}$ over $\mathbb{U}$, their aperiodic cross-correlation function is defined as*

$$\rho_{\mathbf{a},\mathbf{b}}(\tau)=\sum_{k=0}^{N-1-\tau} a_k b_{k+\tau}, \quad 0 \leq \tau \leq N-1.$$

When $\mathbf{a}=\mathbf{b}$, $\rho_{\mathbf{a},\mathbf{b}}(\tau)$ is called aperiodic autocorrelation function (AACF) of a and is denoted as $\rho_{\mathbf{a}}(\tau)$.

Definition 2. *A pair of sequences $(\mathbf{a},\mathbf{b})$, each of length N, is said to be a Golay complementary pair (GCP), if and only if*

$$\rho_{\mathbf{a}}(\tau)+\rho_{\mathbf{b}}(\tau)=0, \text{ for all } 1 \leq \tau \leq N-1.$$

Definition 3. *A pair of sequences $(\mathbf{a},\mathbf{b})$, each of length N, is said to be a Z-complementary pair (ZCP) with zero correlation zone (ZCZ) width Z, if and only if*

$$\rho_{\mathbf{a}}(\tau)+\rho_{\mathbf{b}}(\tau)=0, \text{ for all } 1 \leq \tau \leq Z-1.$$

When N is even, the ZCP is called an EB-ZCP. If $Z=N$, this pair $(\mathbf{a},\mathbf{b})$ is a GCP.

Definition 4. *A GCP $(\mathbf{c},\mathbf{d})$ is called a complementary mate of a GCP $(\mathbf{a},\mathbf{b})$, if*

$$\rho_{\mathbf{a},\mathbf{c}}(\tau)+\rho_{\mathbf{b},\mathbf{d}}(\tau)=0, \text{ for all } 0 \leq \tau \leq N-1.$$

A useful lemma in [22] is given below.

Lemma 1. *Let $(\mathbf{a},\mathbf{b})$ be a GCP, then $(\mathbf{c},\mathbf{d})=(\overleftarrow{\mathbf{b}},-\overleftarrow{\mathbf{a}})$ is complementary mate of GCP $(\mathbf{a},\mathbf{b})$.*

3. New Construction of Even-Length Binary Z-Complementary Pairs

In this section, we give a systematic construction of EB-ZCPs.

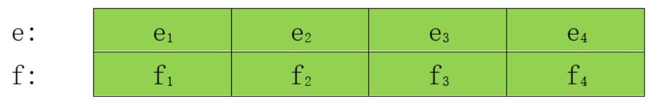

Fig. 1 Divide the sequences $\mathbf{e},\mathbf{f}$.

3.1 Systematic Construction

Construction 1. *Step 1: Let $(\mathbf{x},\mathbf{y})$ be a GCP of length $N=2^{\alpha}10^{\beta}26^{\gamma}$. Construct*

$$\mathbf{a}=[\mathbf{x},\mathbf{y},\mathbf{x},-\mathbf{y}],$$
$$\mathbf{b}=[\mathbf{x},\mathbf{y},-\mathbf{x},\mathbf{y}].$$

Step 2: Set $(\mathbf{c},\mathbf{d})=(\overleftarrow{\mathbf{b}},-\overleftarrow{\mathbf{a}})$, and $\mathbf{e}=[\mathbf{a},\mathbf{c}]$, $\mathbf{f}=[\mathbf{b},\mathbf{d}]$.

Step 3: Sequences $\mathbf{e},\mathbf{f}$ are divided into 4 parts $\mathbf{e_1},\mathbf{e_2},\mathbf{e_3},\mathbf{e_4}$ and $\mathbf{f_1},\mathbf{f_2},\mathbf{f_3},\mathbf{f_4}$ on average, as shown in Fig. 1.

Step 4: Construct two sequences $\mathbf{p}$ and $\mathbf{q}$

$$\mathbf{p}=[x_1,\mathbf{e_1},x_2,\mathbf{e_2},\mathbf{e_3},x_3,\mathbf{e}_4,x_4],$$
$$\mathbf{q}=[y_1,\mathbf{f}_1,y_2,\mathbf{f_2},\mathbf{f}_3,y_3,\mathbf{f_4},y_4],$$

where

$$\begin{pmatrix} x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \end{pmatrix}=\begin{pmatrix} -1 & 1 & 1 & 1 \\ -1 & -1 & 1 & -1 \end{pmatrix}. \tag{1}$$

The construction of sequences $\mathbf{p},\mathbf{q}$ is illustrated in Fig. 2.

As the Step 1 in Construction 1, it is easy to verify the following property.

Property 1. *Suppose that $(\mathbf{a},\mathbf{b})$ is a GCP generated by Step 1 in Construction 1. Write $\mathbf{a}=(a_0,a_1,\ldots,a_{4N-1})$ and $\mathbf{b}=(b_0,b_1,\ldots,b_{4N-1})$, then we have*

$$\begin{aligned} &a_k=b_k, \text{ when } k \in \{0,1,\cdots,2N-1\}; \\ &a_k=-b_k, \text{ when } k \in \{2N,2N+1,\cdots,4N-1\}; \\ &a_k=a_{k+2N}, \text{ when } k \in \{0,1,\cdots,N-1\}; \\ &a_k=-a_{k+2N}, \text{ when } k \in \{N,N+1,\cdots,2N-1\}. \end{aligned} \tag{2}$$

The following result follows directly from the properties of GCPs.

Lemma 2. *Let sequences $\mathbf{e_1},\mathbf{e_2},\mathbf{e_3},\mathbf{e_4},\mathbf{f_1},\mathbf{f_2},\mathbf{f_3},\mathbf{f_4}$ be generated by Step 3 of Construction 1. Then we have*

1. *Both $(\mathbf{e}_i,\mathbf{e}_{i+1})(i=1,3)$ and $(\mathbf{e}_i,\mathbf{e}_{i+2})(i=1,2)$ are*

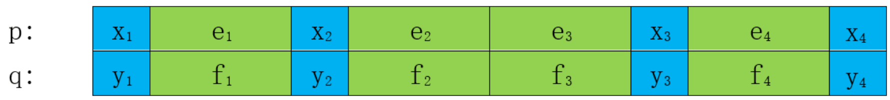

Fig. 2 Construction of sequence $\mathbf{p}, \mathbf{q}$.

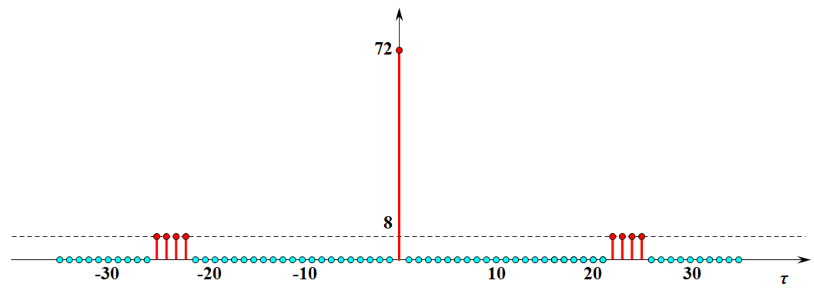

Fig. 3 AACF sum magnitudes of EB-ZCP.

GCPs;

2. $(\mathbf{e}_1, \mathbf{e}_2)$ *is a Golay mate of* $(\mathbf{e}_3, \mathbf{e}_4)$*;*
$(\mathbf{f}_1, \mathbf{f}_2)$ *is a Golay mate of* $(\mathbf{f}_3, \mathbf{f}_4)$*;*
$(\mathbf{e}_1, \mathbf{e}_3)$ *is a Golay mate of* $(\mathbf{e}_2, \mathbf{e}_4)$*;*
$(\mathbf{f}_1, \mathbf{f}_3)$ *is a Golay mate of* $(\mathbf{f}_2, \mathbf{f}_4)$.

Our main result is the following.

Theorem 1. *For the sequence pair* $(\mathbf{p}, \mathbf{q})$ *generated by Construction 1, it is an EB-ZCP of length* $8N+4$ *with ZCZ width* $Z = 5N + 2$*. Furthermore,*

$$\rho_{\mathbf{p}}(\tau) + \rho_{\mathbf{q}}(\tau) = \begin{cases} 2(8N+4), & \tau = 0; \\ 0, & 0 < \tau < Z; \\ \pm 8, & Z \le \tau \le 6N+1; \\ 0, & \textit{otherwise}. \end{cases}$$

Proof. See Appendix. □

Remark 1. *In [20], Adhikary et al. constructed ZCPs with the asymptotic ZCZ ratio* $3/4$ *by inserting two points to a Golay mate. Besides, by using the Kronecker product method [14] and the ZCPs from [20], ZCPs of length* $8N+4$ *and with ZCZ width* $6N+2$ *can be obtained. While in this paper, inspired by the work in [20], by inserting four points to a Golay mate, the above theorem gives new ZCPs with* $5/8$ *ZCZ ratio asymptotically.*

3.2 PAPR of the Proposed ZCPs

Like GCPs, one application of ZCPs is the reduction of peak-to-mean envelop power ratio of multicarrier signals [4], [19]. In this section, we discuss the PAPR of the proposed ZCPs.

Let $\mathbf{a} = (a_0, a_1, \ldots, a_{L-1})$ be the corresponding BPSK modulated sequence over $\mathbb{U}$. The time-domain baseband OFDM signal can be written as

$$S_{\mathbf{a}}(t) = \sum_{i=0}^{L-1} a_i e^{j2\pi i t}, \quad 0 \le t \le 1, \tag{3}$$

Table 2 PAPR of SOME EB-ZCPs.

length of seed sequence **x** or **y**	length of ZCP **p** or **q**	PAPR(**p**)	PAPR(**q**)
1	12	3	3.1986
2	20	2.2654	3.3339
4	36	2.7778	2.6652
8	68	2.4572	2.4905
10	84	2.6110	2.6983
20	164	2.4001	2.4804
26	212	2.3683	2.4290
52	420	2.3307	2.2613
260	2084	2.1398	2.1527

The data in above table comes from part of the sequence, not all of it.

where the number of subcarriers is equal to length of sequence $\mathbf{a}$. Denote $P_{\mathbf{a}}(t) = |S_{\mathbf{a}}(t)|^2$ as the instantaneous power and P_{av} as the average power. For a BPSK modulated sequence, we have P_{av} is L and hence the PAPR of a sequence $\mathbf{a}$ is given by

$$\mathrm{PAPR}(\mathbf{a}) = \max_{0\le t\le 1} \frac{P_{\mathbf{a}}(t)}{P_{av}} = \max_{0\le t\le 1} \frac{P_{\mathbf{a}}(t)}{L}.$$

By (3), we have

$$P_{\mathbf{a}}(t) = \rho_{\mathbf{a}}(0) + 2\Re e\left(\sum_{\tau=1}^{L-1} \rho_{\mathbf{a}}(\tau) e^{-\tau j2\pi t}\right).$$

Let $(\mathbf{a}, \mathbf{b})$ be two binary sequences of length L, then we have

$$\mathrm{PAPR}(\mathbf{a}) \le 2 + \frac{2}{L}\sum_{\tau=1}^{L-1} |\rho_{\mathbf{a}}(\tau) + \rho_{\mathbf{b}}(\tau)|. \tag{4}$$

The following result follows directly from (4).

Theorem 2. *Suppose that* $(\mathbf{p}, \mathbf{q})$ *is a ZCP generated by Construction 1. Then sequence* $\mathbf{p}$ *or* $\mathbf{q}$ *has a PAPR less than 4.*

Table 2 lists the PAPR of some ZCPs generated by Construction 1. It can be seen that the actual PAPR of these sequences are smaller than 3 when the length of sequence pair is larger than 4. It might be possible to derive a better bound for the PAPR of the proposed sequences.

3.3 An Example

The following example will illustrate the proposed construction step by step.

Step 1: Let $x = [+++-], y = [+-++]$, then $(\mathbf{a}, \mathbf{b})$ is a GCP of length 16. The first $2^{4-1} = 8$ columns of $(\mathbf{a}, \mathbf{b})$ has entries with identical signs in each column.

$$\begin{pmatrix} \mathbf{a} \\ \mathbf{b} \end{pmatrix} = \begin{pmatrix} +++-+-+++++--+-- \\ +++-+-++---++-++ \end{pmatrix}.$$

Step 2: Construct $(\mathbf{c}, \mathbf{d})$ as a Golay mate of $(\mathbf{a}, \mathbf{b})$,

$$\begin{pmatrix} \mathbf{c} \\ \mathbf{d} \end{pmatrix} = \begin{pmatrix} ++-++---++-+-+++ \\ ++-++-----+-+--- \end{pmatrix}.$$

And define $\mathbf{e} = [\mathbf{a}, \mathbf{c}], \mathbf{f} = [\mathbf{b}, \mathbf{d}]$.

Step 3: Divide sequence $\mathbf{e}, \mathbf{f}$ into 4 parts on average,

$$\mathbf{e} = (+++-+-++\,+++--+-- \\ ++-++---\,++-+-+++);$$

$$\mathbf{f} = (+++-+-++\,---++-++ \\ ++-++---\,--+-+---);$$

Step 4: Insert four elements in $\mathbf{e}, \mathbf{f}$,

$$\mathbf{p} = (-+++-+-+++++++--+-- \\ ++-++---++++-+-++++);$$

$$\mathbf{q} = (-+++-+-++----++-++ \\ ++-++---+--+-+----);$$

Then $(\mathbf{p}, \mathbf{q})$ is a length-36 EB-ZCP with a ZCZ width of 22 because

$$\left(\rho_{\mathbf{p}}(\tau) + \rho_{\mathbf{q}}(\tau)\right)_{\tau=0}^{35} = (72, \mathbf{0}_{21}, 8_4, \mathbf{0}_{10}).$$

The AACP sum magnitudes of $(\mathbf{p}, \mathbf{q})$ are shown in Fig. 3. Furthermore, the PAPR of sequences $\mathbf{p}, \mathbf{q}$ are 2.9018, 2.9415, respectively.

4. Conclusion

In this paper, a construction of EB-ZCPs was proposed. The key of the proposed construction is to use some interesting intrinsic structure properties of binary GCPs. The proposed EB-ZCPs have lengths $2^{\alpha+3}10^{\beta}26^{\gamma} + 4$ and ZCZ widths $5 \cdot 2^{\alpha}10^{\beta}26^{\gamma} + 2$. The PAPR of the proposed sequences was proved to upper bounded by 4.

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Appendix: Proof of Theorem 1

We distinguish among the following eight cases to calculate the aperiodic autocorrelation sums.

Case 1. $1 \le \tau \le 2N$. We have

$$\begin{aligned}\rho_{\mathbf{p}}(\tau) =& x_1 a_{\tau-1} + \rho_{\mathbf{e}_1}(\tau) + x_2 a_{2N-\tau} + x_2 a_{2N+\tau-1} + \rho_{\mathbf{e}_2}(\tau) \\ &+ \rho_{\mathbf{e}_2\mathbf{e}_3}(\tau) + \rho_{\mathbf{e}_3}(\tau) + x_3 c_{2N-\tau} + x_3 c_{2N+\tau-1} \\ &+ \rho_{\mathbf{e}_4}(\tau) + x_4 c_{4-\tau} \\ =& x_1 a_{\tau-1} + x_2 \left(a_{2N-\tau} + a_{2N+\tau-1}\right) \\ &+ x_4 c_{4N-\tau} + x_3 \left(c_{2N-\tau} + c_{2N+\tau-1}\right) \\ &+ \rho_{\mathbf{e}_1}(\tau) + \rho_{\mathbf{e}_2}(\tau) + \rho_{\mathbf{e}_3}(\tau) + \rho_{\mathbf{e}_4}(\tau) + \rho_{\mathbf{e}_2 e_3}(\tau).\end{aligned}$$

$$\begin{aligned}\rho_{\mathbf{q}}(\tau) =& y_1 b_{\tau-1} + \rho_{\mathbf{f}_1}(\tau) + y_2 b_{2N-\tau} + y_2 b_{2N+\tau-1} + \rho_{\mathbf{f}_2}(\tau) \\ &+ \rho_{\mathbf{f}_2\mathbf{f}_3}(\tau) + \rho_{\mathbf{f}_3}(\tau) + y_3 d_{2N-\tau} + y_3 d_{2N+\tau-1} \\ &+ \rho_{\mathbf{f}_4}(\tau) + y_4 d_{4N-\tau} \\ =& y_1 b_{\tau-1} + y_2 \left(b_{2N-\tau} + b_{2N+\tau-1}\right) \\ &+ y_4 d_{4N-\tau} + y_3 \left(d_{2N-\tau} + d_{2N+\tau-1}\right) \\ &+ \rho_{\mathbf{f}_1}(\tau) + \rho_{\mathbf{f}_2}(\tau) + \rho_{\mathbf{f}_3}(\tau) + \rho_{\mathbf{f}_4}(\tau) + \rho_{\mathbf{f}_2\mathbf{f}_3}(\tau).\end{aligned}$$

Hence, we have

$$\begin{aligned}&\rho_{\mathbf{p}}(\tau) + \rho_{\mathbf{q}}(\tau) = x_1 a_{\tau-1} \\ &\quad + x_2 \left(a_{2N-\tau} + a_{2N+\tau-1}\right) + x_3 \left(c_{2N-\tau} + c_{2N+\tau-1}\right) \\ &\quad + x_4 c_{4N-\tau} + y_1 b_{\tau-1} + y_2 \left(b_{2N-\tau} + b_{2N+\tau-1}\right) \\ &\quad + y_3 \left(d_{2N-\tau} + d_{2N+\tau-1}\right) + y_4 d_{4N-\tau} + \rho_{\mathbf{e}_1}(\tau) + \rho_{\mathbf{e}_2}(\tau) \\ &\quad + \rho_{\mathbf{e}_3}(\tau) + \rho_{\mathbf{e}_4}(\tau) + \rho_{\mathbf{f}_1}(\tau) + \rho_{\mathbf{f}_2}(\tau) + \rho_{\mathbf{f}_3}(\tau) + \rho_{\mathbf{f}_4}(\tau).\end{aligned}$$

By Lemma 2, $\rho_{\mathrm{e}_1}(\tau)+\rho_{\mathrm{e}_2}(\tau) = 0$, $\rho_{\mathrm{e}_3}(\tau)+\rho_{\mathrm{e}_4}(\tau) = 0$, $\rho_{\mathrm{f}_1}(\tau)+\rho_{\mathrm{f}_2}(\tau) = 0$, $\rho_{\mathrm{f}_3}(\tau) + \rho_{\mathrm{f}_4}(\tau) = 0$. Besides, it is easy to see that $\mathbf{e}_2 = -\mathbf{f}_2$ and $\mathbf{e}_3 = \mathbf{f}_3$. Hence, we have $\rho_{e_2 e_3}(\tau)+\rho_{f_2 f_3}(\tau) = 0$.

It then follows that

$$\begin{aligned}&\rho_{\mathbf{p}}(\tau) + \rho_{\mathbf{q}}(\tau) = x_1 a_{\tau-1} \\ &\quad + x_2 \left(a_{2N-\tau} + a_{2N+\tau-1}\right) + x_3 \left(c_{2N-\tau} + c_{2N+\tau-1}\right) \\ &\quad + x_4 c_{4N-\tau} + y_1 b_{\tau-1} + y_2 \left(b_{2N-\tau} + b_{2N+\tau-1}\right) \\ &\quad + y_3 \left(d_{2N-\tau} + d_{2N+\tau-1}\right) + y_4 d_{4N-\tau}.\end{aligned}$$

According to Property 1 and Eq. (1), we have

$$\begin{aligned}\rho_{\mathbf{p}}(\tau) + \rho_{\mathbf{q}}(\tau) &= -2a_{\tau-1} + 2a_{2N+\tau-1} + 2c_{2N-\tau} + 2c_{4N} \\ &= -2a_{\tau-1} + 2a_{2N+\tau-1} + 2b_{2N+\tau-1} + 2b_{\tau-1} \\ &= 0.\end{aligned}$$

Case 2. $\tau = 2N + 1$. We have

$$\begin{aligned}\rho_{\mathbf{p}}(\tau) =& x_1 x_2 + \rho_{\mathbf{e}_1\mathbf{e}_2}(0) + x_2 c_0 + \rho_{\mathbf{e}_2\mathbf{e}_3}(1) \\ &+ x_3 a_{4N-1} + \rho_{\mathbf{e}_3\mathbf{e}_4}(0) + x_3 x_4 \\ =& x_1 x_2 + x_3 x_4 + x_2 c_0 + x_3 a_{N-1} \\ &+ \rho_{\mathbf{e}_1\mathbf{e}_2}(0) + \rho_{\mathbf{e}_3\mathbf{e}_4}(0) + \rho_{\mathbf{e}_2\mathbf{e}_3}(1).\end{aligned}$$

By Lemma 2, we have

$$\rho_{\mathbf{e}_1\mathbf{e}_2}(0) = \rho_{\mathbf{e}_3\mathbf{e}_4}(0) = 0.$$

Hence,

$$\rho_{\mathbf{p}}(\tau) = x_1 x_2 + x_3 x_4 + x_2 c_0 + x_3 a_{4N-1} + \rho_{\mathbf{e}_2\mathbf{e}_3}(1).$$

Similarly,

$$\rho_{\mathbf{q}}(\tau) = y_1 y_2 + y_3 y_4 + y_2 d_0 + y_3 b_{4N-1} + \rho_{\mathbf{f}_2 f_3}(1).$$

It follows from Lemma 2 and (1) that

$$\begin{aligned}\rho_{\mathbf{p}}(\tau) + \rho_{\mathbf{q}}(\tau) &= x_2 c_0 + x_3 a_{4N-1} + y_2 d_0 + y_3 b_{4N-1} \\ &= \left(x_3 - x_2\right) a_{4N-1} + \left(y_2 + y_3\right) b_{4N-1} \\ &= 0.\end{aligned}$$

Similarly, we have the AACFs are zero following cases:

- Case 3. $2N + 2 \le \tau \le 4N$;
- Case 4. $\tau = 4N + 1$;
- Case 6. $\tau = 6N + 2$;
- Case 7. $6N + 3 \le \tau \le 8N + 2$;
- Case 8. $\tau = 8N + 3$.

Besides, we should discuss Case 5 separately.

Case 5. $4N + 2 \le \tau \le 6N + 1$. We have

$$\begin{aligned}\rho_{\mathbf{p}}(\tau) =& x_1 c_{\tau-4N-2} \\ &+ \rho_{\mathbf{e}_1\mathbf{e}_3}(\tau - 4N - 1) + \rho_{\mathbf{e}_2\mathbf{e}_4}(\tau - 4N - 1) \\ &+ x_3 a_{6N+1-\tau} + x_2 c_{\tau-2N-2} + x_4 a_{8N+1-\tau}, \\ \rho_{\mathbf{q}}(\tau) =& y_1 d_{\tau-4N-2} \\ &+ \rho_{\mathbf{f}_1\mathbf{f}_3}(\tau - 4N - 1) + \rho_{\mathbf{f}_2\mathbf{f}_4}(\tau - 4N - 1) \\ &+ y_3 b_{6N+1-\tau} + y_2 d_{\tau-2N-2} + y_4 b_{8N+1-\tau}.\end{aligned}$$

According to Lemma 2 and (1), we have

$$\begin{aligned}&\rho_{\mathrm{P}}(\tau) + \rho_{\mathrm{q}}(\tau) = 4a_{6N+1-\tau} + 4a_{8N+1-\tau} \\ &\quad - c_{\tau-4N-2} + c_{\tau-2N-2} + a_{6N+1-\tau} + a_{8N+1-\tau} \\ &\quad - d_{\tau-4N-2} - d_{\tau-2N-2} + b_{6N+1-\tau} - b_{8N+1-\tau} \\ &= 4a_{6N+1-\tau} + 4a_{8N+1-\tau}.\end{aligned}$$

Using Property 1, we have

$$\begin{aligned}\rho_{\mathbf{p}}(\tau) + \rho_{\mathbf{q}}(\tau) &= 4a_{6N+1-\tau} + 4a_{8N+1-\tau} \\ &= \begin{cases} 0, & 4N + 2 \le \tau \le 5N + 1; \\ \pm 8, & 5N + 2 \le \tau \le 6N + 1. \end{cases}\end{aligned}$$

Summarizing the eight cases above completes the proof of this theorem.